

QUOTIENTS BY $C^* \times C^*$ ACTIONS

BY

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ABSTRACT. Let $T \approx C^* \times C^*$ act meromorphically on a compact Kähler manifold X , e.g. algebraically on a projective manifold. The following is a basic question from geometric invariant theory whose answer is unknown even if X is projective.

PROBLEM. *Classify all T -invariant open subsets U of X such that the geometric quotient $U \rightarrow U/T$ exists with U/T a compact complex space (necessarily algebraic if X is).*

In this paper a simple to state and use solution to this problem is given. The classification of U is reduced to finite combinatorics. Associated to the T action on X is a certain finite 2-complex $\mathcal{C}(X)$. Certain $\{0, 1\}$ valued functions, called moment measures, are defined in the set of 2-cells of $\mathcal{C}(X)$. There is a natural one-to-one correspondence between the U with compact quotients, U/T , and the moment measures.

Let X be a connected compact Kähler manifold with a meromorphic action of $T \approx (C^*)^k$:

$$T \times X \rightarrow X.$$

The following is a basic question from geometric invariant theory whose answer is unknown for general k even if X is projective.

PROBLEM. *Classify all T -invariant open subsets U of X such that the geometric quotient (cf. §0 for definitions) $U \rightarrow U/T$ exists with U/T a compact complex space.*

In this paper we give a simple to state and use solution to this problem when $k = 2$.

The previous work on this problem consists of two parts:

(a) as a special case of his geometric invariant theory, D. Mumford [15] gave a prescription for construction of some of those geometric quotients with U/T projective,

(b) for $k = 1$ a complete and simple answer was given in [2] for very general X by the authors of this paper [2]. This answer was in terms of a certain graph with a finite set of vertices (see also [3] for the algebraic case).

It is notable that even for simple manifolds such as Grassmannians there exist geometric quotients $U \rightarrow U/T$ of Zariski open U with complete nonprojective schemes as quotients. Moreover Mumford's description of U with projective quotients using stability is redundant; more than one ample line bundle and more than one T linearization of the same line bundle can give the same open set U . For the

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above reasons it is not at all clear from Mumford's theory whether the set of all U with a compact (or even projective) geometric quotient has a simple description.

The geometric idea underlying our work is simple. It rests on three principles.

The first is that for a T -invariant open subset $U \subseteq X$ to have a compact quotient it is necessary and sufficient that given any "limit", W , of a sequence of closures of "generic" orbits of T , $W \cap U = Tx$ for some x with $\dim Tx = k$. This is made precise by using the work of Fujiki and Lieberman on Douady spaces of closures of orbits (see §0).

The second principle is that the existence of U/T as a compact analytic space should have a cohomological interpretation, i.e. there should exist an analogue of the Thom class of a vector bundle. This is carried out in §1.

The third principle is that the essential geometry of the Douady space of closures of orbits of T should have a finite combinatorial description as a k -dimensional cell complex where $\dim T = k$. Here the work of Atiyah (see §0) makes this relatively straightforward for X smooth and Kähler.

This combinatorial object $\mathcal{C}(X)$ which is related to torus embeddings is described in §2 for $\dim T = 2$. The "Thom class" mentioned in the last paragraph survives as a $\{0, 1\}$ valued function μ on the cells of $\mathcal{C}(X)$, and is called a "moment measure".

Our main theorem says that there is a one-to-one correspondence between the T -invariant open U with compact quotients and the moment measures.

For $\dim T \geq 3$ it is much more difficult to define and very messy to manipulate the combinatorial polyhedron $\mathcal{C}(X)$. Nonetheless we would do so if we could prove our main result in that generality. The easier half of the proof of our main result, which says that moment measures give rise to T -invariant open U with compact geometric quotients U/T , does generalize. Unfortunately, when $\dim T \geq 3$, we cannot prove the more important converse. The difficulties come in §3 where we construct certain homology classes by transcendental methods. To prove the general converse we need to know about the Hodge types of the analogues of these classes. This seems to require new ideas.

Let us describe the contents of our paper in detail.

Let X be a compact Kähler manifold with a meromorphic action of an analytic torus $T = (\mathbb{C}^*)^k$. Let U be an open T -invariant subset of X . By the quotient U/T we mean the geometric quotient (orbit space) of U ; if U/T exists it is an analytic set (and hence the underlying topological space is Hausdorff).

In §0 we gather analytic results in a suitable form basic for our study. Most important are the theorems of Fujiki and Lieberman on Douady families of orbits (0.2.2) and the results of Atiyah on moment functions ((0.5.1); (0.5.2)).

§1 contains some topological results concerning group actions and orbit spaces. Fundamental for further applications in the paper are: Theorem (1.3) which gives a necessary and sufficient condition for compactness of the orbit space U/T in terms of cohomological properties of the space U , Theorem (1.5) and Corollary (1.7.1) which relate the rational cohomology of those U for which U/T is compact and of X , and Theorem (1.8) which shows that the first rational homology group of a compact Kähler manifold X is isomorphic to the first rational homology group of

the source of X for any action of C^* on X . All these results except Corollary (1.7.1) and Theorem (1.8) are proved for a k -dimensional torus T , where $k \geq 1$.

The main aim of §2 is to describe for a given action of a two-dimensional torus T on X , the two-dimensional oriented cell complex $\mathcal{C}(X)$ that we use to classify quotients by T . This finite combinatorial object is defined in (2.1) and (2.8). The vertices of the complex are represented by connected components of the fixed point set X^T . One- and two-dimensional cells are some ordered subset of the set of vertices corresponding to different types of one- and two-dimensional orbits. The properties of the complex that are basic for our work are derived by using moment functions defined on X . The combinatorial structure of the complex allows us to define a notion of a subdivision of a two-dimensional cell (2.6) which should be considered as a combinatorial counterpart of a geometric notion of degeneration of a type of orbits when passing to the limit in the Douady space (see (0.5.1)(b)). Finally the notion of a subdivision allows us to introduce the notion of a moment measure (2.13). A moment measure is a $\{0, 1\}$ valued function defined on the set of 2-cells of $\mathcal{C}(X)$ which is additive with respect to subdivisions. Points whose orbits determine cells of μ -measure 1 are called μ -stable.

Theorem (3.1), the main result of §3 (and of the paper), says that open T -invariant subsets U of X such that U/T exists and is compact are exactly the sets of μ -stable points for moment measures μ . The main obstruction to generalizing the theorem to the case where $\dim T = k > 2$ is the lack of proper generalizations of some of the above results concerning topological properties of actions of many-dimensional tori. We end §3 with some simple examples that we hope will help with understanding the main result.

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0. Notation and background material. In this section we set up the notation that we will use and collect some results that we will need.

(0.1) T will always denote the k -dimensional torus $(C^*)^k$. T has an Iwasawa decomposition $T \approx K \times A$, where $K \approx (S^1)^k$ is the maximal compact subgroup of T and where $A \approx (\mathcal{R}^+)^k$. The character group of T is denoted by $\chi(T)$ and the group of one-parameter subgroups of T is denoted by $\chi^*(T)$. The map

$$\chi(T) \times \chi^*(T) \xrightarrow{\Gamma} \mathcal{Z},$$

defined by $\Gamma(\chi, \alpha) = \chi \circ \alpha \in \chi(C^*) = \mathcal{Z}$, is a perfect pairing. Since $T \approx (C^*)^k$ we have $\chi(T) \approx \mathcal{Z}^k$ and $\chi^*(T) \approx \mathcal{Z}^k$. Then $\chi_{\mathcal{R}}(T) = \chi(T) \otimes_{\mathcal{Z}} \mathcal{R}$ and $\chi_{\mathcal{R}}^*(T) = \chi^*(T) \otimes_{\mathcal{Z}} \mathcal{R}$ are dual k -dimensional vector spaces. Using the fixed isomorphism $T \approx (C^*)^k$ we have a canonical basis $\{e_1, \dots, e_k\} \subseteq \chi(T)$ with $e_1 = (1, 0, \dots, 0), \dots, e_k = (0, \dots, 0, 1)$.

Using this choice of basis we get orientations on $\chi_{\mathcal{R}}(T)$ and $\chi_{\mathcal{R}}^*(T)$.

(0.2) X will always denote a compact connected Kähler manifold with a holomorphic action of T . It is further assumed that the action is meromorphic, i.e. that

$T \times X \rightarrow X$ extends to a meromorphic map $(\mathcal{P}_C^1)^k \times X \rightarrow X$. It is a result of Sommesse [16] that this is equivalent to $X^T \neq \emptyset$, where X^T denotes the fixed point set of the action. Let $X^T = F_1 \cup \cdots \cup F_r$ be the decomposition of X^T into connected components. Given any one-parameter subgroup $\alpha: \mathbf{C}^* \rightarrow T$, X^α denotes the fixed point set of the induced action of \mathbf{C}^* on X .

We need a few results on orbit structure.

(0.2.1) LEMMA. *Let X and T be as above. Then $\{x | \dim Tx < k\}$ is a closed analytic subset of X .*

PROOF. Consider $m: T \times X \rightarrow X \times X$ given by $(t, x) \rightarrow (tx, x)$. Let $m_{m^{-1}(\Delta)}: m^{-1}(\Delta) \rightarrow \Delta$ be the induced map where Δ is the diagonal of $X \times X$. Under the natural identification of X with Δ , the set $\{x | \dim Tx < k\}$ coincides with the set $\{x | \dim m_{m^{-1}(\Delta)}^{-1}(x) > 0\}$. By semicontinuity of dimension in the Zariski topology, the lemma follows. \square

The next two results are immediate consequences of the work of A. Fujiki [8, 9, 10] and D. Lieberman [13] on compactness of components of the Douady space of Kähler manifolds.

(0.2.2) THEOREM. *Let X and T be as above. There is for any $x \in X$ with $\dim Tx = k$ a diagram*

$$(*) \quad \begin{array}{ccc} Z_x & \xrightarrow{\phi_x} & X \\ f_x \downarrow & & \\ Q_x & & \end{array}$$

with the following properties:

- (a) f_x is a flat surjective morphism of connected compact complex spaces Z_x and Q_x .
- (b) the restriction of ϕ_x to each fibre $Z_x(q) = f_x^{-1}(q)$ is an embedding and there is a $q \in Q$ such that $\phi_x(Z_x(q)) = \overline{Tx}$,
- (c) there is a natural action of T on Z_x making f_x and ϕ_x equivariant with respect to the trivial action of T on Q_x and the given action of T on X .
- (d) there is a dense Zariski open set $\mathcal{O}_x \subseteq Q_x$ such that for each $q \in \mathcal{O}_x$, $Z_x(q)$ is reduced and $\phi_x(Z_x(q))$ is the closure of a T -orbit,
- (e) the reduction of every fibre of f_x is pure k -dimensional and for fibres $\{Z_x(q), Z_x(q')\}$ that are reduced, $\phi_x(Z_x(q)) = \phi_x(Z_x(q'))$ only if $q = q'$,
- (f) given any diagram

$$(**) \quad \begin{array}{ccc} Z' & \xrightarrow{\phi'} & X \\ f' \downarrow & & \\ Q' & & \end{array}$$

that satisfies properties (a) through (e) there is a holomorphic map $c: Q' \rightarrow Q_x$ such that $(**)$ is the pullback of $(*)$.

PROOF. Let \mathcal{D}_x be the connected component of the Douady space of X which contains the point corresponding to \overline{Tx} . By the defining properties of the Douady space there is a diagram

$$\begin{array}{ccc} & & \Phi \\ & & \rightarrow \\ \mathcal{H}_x & & X \\ & & \downarrow \\ & & F \\ & & \downarrow \\ & & \mathcal{D}_x \end{array}$$

(#)

where \mathcal{H}_x is the family of subspaces of X corresponding to points of \mathcal{D}_x such that:

- (a) F is a flat surjection and Φ is an embedding on any fibre of F ,
- (b) there is a point $q \in \mathcal{D}_x$ such that $\Phi(F^{-1}(q)) = \overline{Tx}$,
- (c) given any diagram of connected analytic spaces \mathcal{H} and \mathcal{D} :

$$\begin{array}{ccc} & & \Phi' \\ & & \rightarrow \\ \mathcal{H} & & X \\ & & \downarrow \\ & & F' \\ & & \downarrow \\ & & \mathcal{D} \end{array}$$

(# #)

such that (a) and (b) with Φ' and F' in place of Φ and F respectively are true for this diagram, then there is a holomorphic map $c: \mathcal{D} \rightarrow \mathcal{D}_x$ which induces (# #) from (#).

T acts naturally on \mathcal{D}_x and \mathcal{H}_x . Both actions are holomorphic. Let Q_x be the connected component of \mathcal{D}_x^T that contains the point corresponding to \overline{Tx} . Let $Z_x = F^{-1}(Q_x)$ and let $F_{Z_x} = f_x$, $\Phi_{Z_x} = \phi_x$. By the results of Fujiki [8, 9, 10] and Lieberman [13] \mathcal{D}_x and Q_x are compact. The rest of the proof of (0.2.2) is standard.

□

(0.2.3) By the countability of the number of components the Douady space of a compact Kähler manifold, it follows [8] that there is one diagram from (0.2.2):

$$\begin{array}{ccc} & & \phi_x \\ & & \rightarrow \\ Z_x & & X \\ & & \downarrow \\ & & f_x \\ & & \downarrow \\ & & Q_x \end{array}$$

with Z_x and Q_x compact irreducible spaces and ϕ_x a bimeromorphic holomorphic map. We will drop here the subscripts and refer to the diagram:

$$\begin{array}{ccc} & & \phi \\ & & \rightarrow \\ Z & & X \\ & & \downarrow \\ & & f \\ & & \downarrow \\ & & Q \end{array}$$

(0.2.3*)

(0.2.4) LEMMA. *Only finitely many different diagrams occur in (0.2.2).*

PROOF. By the local compactness result [10] it suffices to bound the volume of any \overline{Tx} with $\dim Tx = k$ with respect to ω^k , where ω is the Kähler class in X . For this it suffices to note that:

- (a) by (0.2.2)(f) and (0.2.3), \overline{Tx} is a component of $\phi(Z(q))$ for some q ,
- (b) two fibres $Z(q)$ and $Z(q')$ of a flat map are homologous,
- (c) the volume of \overline{Tx} is less than or equal to that of $\phi(Z(q))$. □

(0.2.5) COROLLARY. *Let T and X be as above. Let Z_x, Q_x, f_x and ϕ_x be as in (0.2.3). Let V be an irreducible component of the reduction of a fibre $Z_x(q)$ of f_x . Then T has a dense orbit in V .*

PROOF. For simplicity we give the proof for Z, Q, f and ϕ . The only difference between this and the general case are subscripts.

Assume that T has no dense orbit on V . Since $\dim T = \dim V$ by (0.2.2)(e), it follows that each orbit Tx of T for $x \in V$ is of dimension less than $\dim T$. Since T is abelian and since the action is meromorphic, this implies that Tx is left fixed by at least one one-parameter subgroup $C^* \subseteq T$. Since T contains only countably many one-parameter algebraic subgroups of T , it follows from the Baire category theorem that there is a subgroup $T_0 \approx C^*$ of T that fixes all of V .

We will need to work with the analogue of (0.2.3*) for T_0 in place of T . We simply use the subscript 0 when we deal with the T_0 objects, $\{Q_0, Z_0, f_0, \phi_0, \mathcal{O}_0\}$.

Since $\phi_0(f_0^{-1}(\mathcal{O}_0))$ contains a Zariski open subset of X we can choose a sequence of points $\{q_n\} \subseteq \mathcal{O}$ converging to q and such that:

(a) $\mathcal{U}_n = \phi_0(f_0^{-1}(\mathcal{O}_0) - f_0^{-1}(\mathcal{O}_0)^T) \cap \phi(Z(q_n)) \cap \phi(f(\mathcal{O}) - f(\mathcal{O})^T)$ is nonempty for all n .

Since $\phi(Z(q_n))$ converges to $\phi(Z(q))$ in the Hausdorff topology on the space of compact subsets of X , it follows that we can find a sequence $\{x_n\} \subseteq X$ such that:

(b) $x_n \in \mathcal{U}_n$,

(c) $\lim x_n = x \in \phi(V)$ and x lies on no irreducible component of $\phi(Z(q))$ besides $\phi(V)$.

Passing to a subsequence $\{x_n\}$ we can therefore find a sequence $\{y_n\} \subseteq \mathcal{O}_0$ converging to a point $y \in Q_0$ such that $\phi_0(Z_0(y_n)) \supseteq T_0 x_n$. We conclude that $x \in \phi_0(Z_0(y)) \subseteq \phi(Z(q))$. Thus by (c) and the fact that $\phi_0(Z_0(y))$ is connected and uncountable, $\phi_0(Z_0(y))$ meets $\phi(V)$ in an uncountable set. Thus we contradict the fundamental fact that the T_0 fixed point set of $\phi_0(Z_0(y))$ is finite. \square

We need the Kählerian analogue of a result of Sumihiro [17].

(0.3) THEOREM (M. KORAS [12]). *Let X and T be as above. Given any $x \in X^T$ there is a T -invariant Stein neighborhood $\mathcal{U} \subseteq X$ of x and a proper embedding Ψ of \mathcal{U} into C^N such that:*

(a) $\Psi(x) = \text{origin}$,

(b) Ψ is equivariant with respect to the usual action of T on X and a linear action of T on C^N .

This result has the following immediate consequence by pulling back the usual Luna slice theorem on C^N [14].

(0.3.1) COROLLARY. *Let X and T be as above. Let x be a point on X with finite isotropy subgroup $I_x \subseteq T$. Then there exist a neighborhood \mathcal{U} of Tx and an analytic subset D of \mathcal{U} containing x such that $\mathcal{U} \approx T \times_{I_x} D$.*

The above result makes geometric quotients easy to handle.

(0.3.2) THEOREM (HOLMANN [11]). *Let \mathcal{U} be an open T -invariant subset of X . Assume that $\dim Tx = k$ for all $x \in \mathcal{U}$ and that the set \mathcal{U}/T of T orbits on \mathcal{U} is Hausdorff with respect to the induced topology. Then there is a unique complex structure on \mathcal{U}/T such that the quotient map $A: \mathcal{U} \rightarrow \mathcal{U}/T$ is holomorphic and such that given any open set V on \mathcal{U}/T the space of holomorphic functions on V pulls back to the space of holomorphic functions on $A^{-1}(V)$ constant on fibres. \mathcal{U}/T is normal. We call \mathcal{U}/T with this analytic structure, the geometric quotient of \mathcal{U} by T .*

PROOF. The reader can find a proof of the above for $T \approx C^*$ in [2, (0.2)]; there are no real differences between that case and the above case. \square

By W/G , where G is a compact group acting continuously on a complex analytic space W , we mean the space of G orbits with the induced topology. W/G is Hausdorff and has the structure of a simplicial complex if $G \subseteq T$ and the action of G is the restriction of a holomorphic action of T .

The following criteria for existence and compactness of quotients are the analogues of [2, (1.2)]; we use the notation of (0.2.3).

(0.4) THEOREM. *Let $\mathcal{U} \subseteq X$ be a T -invariant open set. A geometric quotient \mathcal{U}/T exists if and only if for each $q \in Q$, $\phi(Z(q)) \cap \mathcal{U}$ is either empty or of the form Tx for some $x \in X$ with $\dim Tx = k$. Assuming that \mathcal{U}/T exists, it is compact if and only if $\phi(Z(q)) \cap \mathcal{U}$ is nonempty for all $q \in Q$.*

PROOF. Assume that $\phi(Z(q)) \cap \mathcal{U} \supseteq Tx \cup Ty$, where Ty does not meet Tx . Since $\phi(f^{-1}(\mathcal{O}) - f^{-1}(\mathcal{O})^T)$ is Zariski open we can find a sequence $q_n \in \mathcal{O}$ such that:

- (a) $\lim q_n = q$,
- (b) $\phi(Z(q_n)) \cap \mathcal{U} = Tx_n$ for some x_n .

Choosing slices D and D' to Tx and Ty it is easy to check that Tx_n converges in \mathcal{U}/T to both Tx and Ty . Thus \mathcal{U}/T is not Hausdorff and the geometric quotient by T does not exist.

Assume that the geometric quotient \mathcal{U}/T exists and that $\phi(Z(q)) \cap \mathcal{U}$ is empty for some q . For the same reason as above there exists a sequence $\{q_n\} \subseteq \mathcal{O}$ such that:

- (a) $\lim q_n = q$,
- (b) $\phi(Z(q_n))$ meets \mathcal{U} .

If \mathcal{U}/T was compact there would be a convergent subsequence $\{y_n\}$ of the sequence of images of $\phi(Z(q_n)) \cap \mathcal{U}$. Let y be the limit point in \mathcal{U}/T and let $x \in \mathcal{U}$ be such that Tx goes to y . Since $\phi(Z(q_n))$ converges in the Hausdorff topology on the compact subsets of X to $\phi(Z(q))$ it follows that $Tx \subseteq \phi(Z(q))$. This contradiction shows that \mathcal{U}/T is not compact.

The other parts of the proof are similar and left to the reader. \square

We need some results of M. Atiyah [1].

(0.5.1) THEOREM. *Let X be a connected compact Kähler manifold on which T acts meromorphically. Then there is a map $\mathcal{J}: X \rightarrow \chi_{\mathcal{A}}(T) \approx \mathcal{R}^k$ that is called the moment map and which is constant on K orbits and connected components of X^T . Further given any Z_x from (0.2.4), we have that:*

- (a) $\mathcal{J}(\phi_x(Z_x)) = \mathcal{J}(\overline{Tx})$ is the convex hull of $\mathcal{J}((Tx) \cap X^T)$, and the last set is the set of vertices, i.e. extreme points of $\mathcal{J}(\overline{Tx})$,

(b) $\not\sim$ factors:

$$\begin{array}{ccc} \phi(Z_x(q)) & \xrightarrow{\not\sim} & \not\sim(\overline{Tx}) \\ a \searrow & & \nearrow b \\ & \phi(Z_x(q))/K & \end{array}$$

with b a homeomorphism for all $q \in Q_x$.

PROOF. By [1] there exists a function $\not\sim$ satisfying all of the above except possibly

$$(*) \quad \not\sim(\phi_x(Z_x)) = \not\sim(\overline{Tx})$$

and (b).

To see $(*)$ note that since there are only a finite number of fixed point components of T in X there are only a finite number of vertices for the convex sets $\not\sim(\overline{Tx'})$, where $x' \in X$. Therefore there are only finitely many possible images $\not\sim(Tx')$ as x' runs over X . Therefore by continuity and the fact that for nearby points q and q' in Q_x , $\phi_x(Z_x(q))$ and $\phi_x(Z_x(q'))$ are nearby in the Hausdorff topology on compact subsets of X , it follows that the set

$$\mathcal{A} = \{ q \in Q_x \mid \not\sim(\phi_x(Z_x(q))) = \not\sim(\overline{Tx}) \}$$

is open.

It is also closed. Take any point $a \in \overline{\mathcal{A}} - \mathcal{A}$. The same argument as above shows that

$$\{ q \in Q_x \mid \not\sim(\phi_x(Z_x(q))) = \not\sim(\phi_x(Z_x(a))) \}$$

is open.

Since Q_x is connected, $(*)$ is proven.

To see (b) note that we have shown above that $\not\sim(\phi_x(Z_x(q))) = \not\sim(\overline{Tx})$. Since $\not\sim$ is constant on K -orbits, it follows that $\not\sim$ factors

$$\begin{array}{ccc} \phi(Z_x(q)) & \xrightarrow{\not\sim} & \not\sim(\overline{Tx}) \\ a \searrow & & \nearrow b \\ & \phi(Z_x(q))/K & \end{array}$$

Since $\phi_x(Z_x(q)) \subseteq \phi(Z(\tilde{q}))$ for some $\tilde{q} \in Q$ it suffices to prove the above result without subscripts. For $\tilde{q} \in \mathcal{O}$ it follows that $\phi(Z(\tilde{q})) = \overline{T\tilde{x}}$, where $\dim T\tilde{x} = k$. For these, b is a homeomorphism by [1]. Since \mathcal{O} is dense in Q we must only show that the set of $q \in Q$, where b fails to be a homeomorphism, is open. To see this assume that b is not a homeomorphism on $\phi(Z(q))/K$. Note that $\phi(Z(q))$ decomposes into a finite number of irreducible components $\{Z_i\}$ on each of which T has a dense k -dimensional orbit by (0.2.5). Thus by [1], b is a homeomorphism in each Z_i/K and maps each to a convex set of $\not\sim(\overline{T\tilde{x}})$. Thus there must be two distinct Z_i which $\not\sim$ maps to a set containing a common open set. Since q near q' implies that $\phi(Z(q))$ is near $\phi(Z(q'))$ in the Hausdorff topology on closed subsets of X and therefore that $\phi(Z(q))/K$ is near $\phi(Z(q'))/K$ in the Hausdorff topology on closed subsets of X/K , it follows that b cannot be a homeomorphism on $\phi(Z(q'))$ for q' near q . \square

(0.5.2) THEOREM. *Let X and T be as above. Given a one-parameter subgroup $\alpha \in \chi^*(T)$, there is a projection $\alpha^*: \chi_{\mathcal{A}}(T) \rightarrow \mathcal{A}$ defined by $\alpha^*(\chi) = \Gamma(\chi, \alpha) = \chi \circ \alpha \in \chi(C^*) = \mathcal{Z}$ for $\chi \in \chi(T)$. Let $\alpha^* \circ \not\!/\!:$ $x \rightarrow \chi_{\mathcal{A}}(\alpha)$ be the composition of α^* and the moment function $\not\!/\!:$. Then for all $x \in X - X^\alpha$, $\alpha^* \circ \not\!/\!(\alpha(t) \cdot x)$ is a monotone increasing function of $|t|$.*

PROOF. This is the assertion on the top of page 8 of [1]. \square

(0.5.3) COROLLARY. *Let X and T be as above. Let $T \approx C^* \times C^*$ and let $\{\alpha, \beta\}$ be two one-parameter subgroups of T that generate T . Let F_α, F_β be connected components of X^α and X^β respectively. Then $\not\!/\!(F_\alpha) \cap \not\!/\!(F_\beta)$ is empty or a single point.*

PROOF. If either F_α or $F_\beta \subseteq X^T$, the result is clear. Therefore assume otherwise. By the above results, $\not\!/\!(F_\alpha)$ and $\not\!/\!(F_\beta)$ are closed line segments. We must therefore only check that they are on different lines.

This is clear since $\alpha^* \circ \not\!/\!(\alpha(t) \cdot x)$ is monotone increasing with $|t|$ and $F_\beta \not\subseteq X^T$. \square

1. Topological results.

(1.0) Throughout this section coefficients are almost always the rational numbers and are therefore suppressed unless different from Q . Thus given a CW complex W , $H_c^j(W)$, $H^j(W)$, $H_j(W)$ denote respectively rational cohomology with compact supports, rational cohomology, and rational homology. As usual $T \approx (C^*)^k = K \times A$ acts meromorphically on a compact connected Kähler manifold X .

(1.1) LEMMA. *Let X and T be as above. Let U be a T -invariant open set such that the geometric quotient U/T exists. Then U is a principal A -bundle and U is homeomorphic to $U/A \times A$ where the product projection to U/A is the quotient map.*

PROOF. Using the slice theorem for T on U and the fact that any isotropy group of a point on U belongs to K , the assertion about U being a bundle over its space of orbits with the induced topology is clear. Choose any continuous section $\sigma: U/A \rightarrow U$; this can be done since A is a cell. The map $U/A \times A \rightarrow U$ given by $(x, a) \rightarrow a \cdot \sigma(x)$ gives the desired homeomorphism. \square

(1.2) LEMMA. *Let T act meromorphically on a compact Kähler manifold X . Let U be a T -invariant open set such that the geometric quotient U/T exists. Let $g: U \rightarrow U/T$ and $g': U/A \rightarrow U/T$ denote the quotient maps. Then the higher direct image sheaves $g_{(j)}(Q)$ and $g'_{(j)}(Q)$ are both constant with stalks isomorphic to $H^j(T)$.*

PROOF. Since A is a cell and $U \rightarrow U/A$ is a principal A -bundle, it follows from the diagram

$$\begin{array}{ccc} U & \rightarrow & U/A \\ g \searrow & & \swarrow g' \\ & U/T & \end{array}$$

and the Leray spectral sequence for a composition of maps that the lemma will follow for g' if it is proved for g .

Given $x \in U$ let D be a slice of Tx through x . By (0.3.1) we have a neighborhood \mathcal{U} of Tx of the form $D \times_{I_x} T$, where I_x is the isotropy group of x ; I_x is the finite subgroup of K .

We have

$$\begin{array}{ccccc} D \times T & \longrightarrow & D \times_{I_x} T & \longrightarrow & (D \times_{I_x} T)/I_x \\ & \searrow & \downarrow g_{\mathcal{U}} & \swarrow \gamma & \\ & & D/I_x \subseteq U/T & & \end{array}$$

Note that γ is a principal T/I_x bundle and thus if D is chosen small enough so that D/I_x is contractible, it follows that γ is a product projection of $\mathcal{U}/I_x \approx D/I_x \times T/I_x$. It follows from the above diagram that

$$H^j(\mathcal{U}) \rightarrow H^j(\text{any fibre of } g_{\mathcal{U}})$$

is an isomorphism. This proves the lemma. \square

(1.3) THEOREM. *Let T act meromorphically on a compact connected Kähler manifold X . Let U be a T -invariant open set such that the geometric quotient U/T exists. Let*

$$a = \min\{j | H_c^j(U) \neq 0\}, \quad b = \min\{j | H_c^j(U/T) \neq 0\}.$$

Then $a = b + k$ and $H_c^a(U) \approx H_c^b(U/T)$. In particular U/T is compact if and only if $a = k$ and $H_c^k(U) \approx Q$.

PROOF. Using (1.1) it follows that

$$(*) \quad H_c^i(U) \approx H_c^{i-k}(U/A)$$

for all i .

Using (1.2), the fact that $U/A \rightarrow U/T$ is proper, and the Leray spectral sequence it follows that

$$(**) \quad H_c^i(U/T) \approx H_c^i(U/A) \quad \text{for } i \leq \min\{j | H_c^j(U/T) \neq 0\}.$$

(*) and (**) together finish the argument. \square

(1.4) In the above we used the map $U \rightarrow A$ to get an isomorphism of $H_c^k(U)$ with Q when U/T is compact. The map $U \rightarrow A$ is unique up to homotopy. This is easily seen since there is a homotopy between any two continuous sections of $U \rightarrow U/A$ that preserves the fibres of $U \rightarrow U/A$. Assume that U/T is compact. From this it follows that the isomorphism $H_c^k(U) \approx Q$ is canonical. It is convenient to refer to the explicit element η_U of $H_c^k(U, \mathcal{Z})$, which is the pullback of the positive generator of $H_c^k(A, \mathcal{Z})$, as the *class of the quotient* U/T . Since $U \rightarrow A$ factors as $U \rightarrow U/K \rightarrow A$ this class is the pullback of a unique class η_V of $H_c^k(V, \mathcal{Z})$ where $V = U/K$. This class is a generalization of the Thom class and has all the functorial properties that one expects. For example, if $x \in U$ or V , then $\overline{Ax} \subseteq X$ or X/K is a 2-chain. Since Ax goes homeomorphically to A under $U \rightarrow A$ or $V \rightarrow A$ it follows that η_U or η_V evaluated on Ax gives 1.

(1.5) THEOREM. *Let X, U , and T be as in (1.3). Then the map $H_c^k(U) \rightarrow H^k(X)$ is the 0 map.*

PROOF. If U/T is noncompact, then by the last theorem $H_c^k(U)$ is 0 and there is nothing to prove. Therefore we can assume without loss of generality that $H_c^k(U) \approx Q$ and that U/T is compact.

Let F be the union of all isotropy groups of all points of U . Since our T action is meromorphic, F is a finite subgroup of K . Then U/F is a principal T/F bundle over U/T .

Let $M = U/F \times_{T/F} \mathcal{P}_C^k$, where T/F acts on \mathcal{P}_C^k with the action

$$t \cdot (z_0, \dots, z_k) \rightarrow (z_0, t_1(t)z_1, \dots, t_k(t)z_k),$$

where $t \in T/F$, $\{t_i\}$ is a basis of $\chi(T/F)$ and $\{z_0, \dots, z_k\}$ are homogeneous coordinates on \mathcal{P}_C^k . Let $\pi: M \rightarrow U/T$ be the induced map. Let Q be a desingularization of U/T . Pulling back π we get a map $\pi': M' \rightarrow Q$. Let $(U/F)'$ denote the pullback of U/F .

The map $U \rightarrow U/F$ extends to a meromorphic map from X to M . Since Q is bimeromorphic to U/T it follows that M is bimeromorphic to M' . Thus letting Z be a desingularization of the graph of the meromorphic map from X to M' gotten by composition, we have the diagram

$$\begin{array}{ccccccc}
 & & U & \rightarrow & X & \xleftarrow{A} & Z \\
 & & \downarrow & & & & \downarrow B \\
 (*) & & U/F & \rightarrow & M & \leftarrow & M' \\
 & & \searrow & & \downarrow \pi & & \downarrow \pi' \\
 & & & & U/T & \leftarrow & Q
 \end{array}$$

Note that if we choose a continuous section σ for the A -bundle $U/F \rightarrow (U/F)/A$, we get continuous sections by pulling back for the bundles $U \rightarrow U/A$, and $(U/F)' \rightarrow (U/F)'/A$. Thus we get a diagram

$$\begin{array}{ccc}
 U & & \\
 \downarrow & & \\
 U/F & \leftarrow & (U/F)' \\
 \swarrow & & \\
 A & &
 \end{array}$$

so that the class η_U of the quotient U/T and the class $\eta_{(U/F)'}$ of the quotient $(U/F)'/(T/F) = Q$ are pullbacks of the same class from $H_c^k(U/F)$.

A basic fact [18] is that A^* and B^* are injective on rational cohomology since A and B are generically finite to one surjective map between compact complex manifolds.

Therefore by $(*)$ and the last two paragraphs it follows that $H_c^k(U) \rightarrow H^k(X)$ is the 0 map if $H_c^k((U/F)') \rightarrow H^k(M')$ is the 0 map.

Since $\pi': M' \rightarrow Q$ is the projectivization of a vector bundle on Q it follows from the Leray-Hirsch theorem that any nonzero element of $H^k(M')$ can be written:

$$(**) \quad \sum_{i=0}^{[k/2]} \omega^i(\pi'^* \eta_i),$$

where $\eta_i \in H^{k-2i}(Q)$ and ω is any class that restricts to a generator of $H^2(\text{fibre of } \pi')$. Since $\eta = \eta_{(U/F)'}$ can be represented as a compactly supported C^∞ form pulled back from A under a projection $(U/F)' \rightarrow A$ associated to a C^∞ section of $(U/F)' \rightarrow Q$, it follows that $\bar{\omega}^i \wedge \eta = 0$ for all $i > 0$ and any C^∞ representative $\bar{\omega}$ of ω . Thus by $(**)$ we get the contradiction that $\eta = \pi'^* \eta_0$, where $\eta_0 \in H^k(Q)$. This implies the theorem. \square

(1.6) Up to this point we have not used the Kähler assumption in any crucial way. Theorem (1.3) holds for any meromorphic action of T on a compact complex space and any T -invariant open $U \subseteq X$ with a geometric quotient U/T . Theorem (1.5) needs in addition that X is a rational homology manifold so that the result of [18] will hold.

The next result is needed for singular varieties.

(1.7) THEOREM. *Let X be a connected compact complex analytic space. Assume that T acts meromorphically on X . Let $g: X \rightarrow X/K$ be the quotient map. The induced map g^* gives an isomorphism $H^1(X/K) \approx H^1(X)$ and an injection $0 \rightarrow H^2(X/K) \rightarrow H^2(X)$.*

PROOF. By the Leray spectral sequence for g and the fact that g is proper with connected fibres, it suffices to prove that $\Gamma(X/K, g_{(1)}(Q)) = 0$. Therefore, assume to the contrary that there is a global nontrivial section, σ of $g_{(1)}(Q)$.

Since g is proper, restriction gives

$$g_{(1)}(X, Q)_y \approx H^1(g^{-1}(y), Q)$$

for the Leray sheaf and all $y \in X/K$. Therefore it suffices to show that σ induces the zero section of $g_{(1)}(\overline{Tx}, Q)$ for all $x \in X$.

Let x be chosen such that $\dim \overline{Tx}$ is as small as possible consistent with σ giving a nontrivial section of $g_{(1)}(\overline{Tx}, Q)$.

Since T is meromorphic it has a fixed point $q \in \overline{Tx}$. Since $g^{-1}(g(q)) = q$, it follows that σ is identically 0 in a neighborhood of $g(q)$ on $g(\overline{Tx})$. Since $Tx \rightarrow (Tx)/K$ is a fibre bundle over a dense open set of \overline{Tx}/K , this implies that σ is zero on \overline{Tx}/K . Therefore there must be an $x' \in \overline{Tx} - Tx$ so that σ gives a nonzero section of $g_{(1)}(\overline{Tx'}, Q)$. Since $Tx' \subseteq \overline{Tx} - Tx$ we get the contradiction that $\dim \overline{Tx'} < \dim \overline{Tx}$. \square

(1.7.1) COROLLARY. *Let X be a compact connected Kähler manifold. Let $T \approx (C^*)^k$ for $k = 1$ or 2 act meromorphically on X . Let U be a T -invariant Zariski open subset of X such that the geometric quotient $U \rightarrow U/T$ exists. Then the inclusion map $H_c^k(U/K) \rightarrow H^k(X/K)$ is the 0 map.*

PROOF. We have the commutative diagram of long exact sequences:

$$\begin{array}{ccccccccccc} \cdots & \rightarrow & H^{k-1}(X) & \rightarrow & H^{k-1}(X-U) & \rightarrow & H_c^k(U) & \rightarrow & H^k(X) & \rightarrow & \cdots \\ & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \\ \cdots & \rightarrow & H^{k-1}(X/K) & \rightarrow & H^{k-1}((X-U)/K) & \rightarrow & H_c^k(U/K) & \rightarrow & H^k(X/K) & \rightarrow & \cdots \end{array}$$

By Theorems (1.5) and (1.7) the result follows immediately. \square

(1.8) THEOREM. *Let X be a compact Kähler manifold with a meromorphic action of $T \approx (C^*)^k$. Let S be the source with respect to some one-parameter subgroups $\alpha \in \chi^*(T)$. Then inclusion i gives an isomorphism $H_1(S/K) \approx H_1(X/K)$.*

PROOF. By Kronecker duality it suffices to prove that $H^1(S/K) \approx H^1(X/K)$.

We have the commutative diagram:

$$\begin{array}{ccc} H^1(S/K) & \leftarrow & H^1(X/K) \\ \downarrow & & \downarrow \\ H^1(S) & \xleftarrow{\mathcal{R}} & H^1(X) \end{array}$$

The vertical arrows are isomorphisms by (1.7.1). The horizontal map \mathcal{R} is an isomorphism by [7, 5, 9]. \square

2. Complexes.

(2.1) Let $\mathcal{C} = \{(V, \text{lin}); E, C\}$ be a system where

- (a) V is a set of elements called vertices,
- (b) lin is a three argument relation on V ; for $\text{lin}(v_1, v_2, v_3)$, read: v_1, v_2, v_3 lie on the same line,
- (c) E is a subset of $V \times V$; elements of E are called (oriented) edges,
- (d) C is a family of cyclically ordered finite subsets of V each containing at least three vertices; the elements of C are called cells.

(2.2) DEFINITION. *A system \mathcal{C} as in (2.1) is called an oriented 2-cell complex (or complex for short):*

- (i) if $\langle v_1, v_2 \rangle \in E$, then $v_1 \neq v_2$,
- (ii) if $\langle v_1, v_2 \rangle \in E$, then $\langle v_2, v_1 \rangle \in E$,
- (iii) if $c \in C$, $v_1, v_2 \in c$ and v_1 is the predecessor of v_2 with respect to the cyclic order of c , then $\langle v_1, v_2 \rangle \in E$,
- (iv) if $c \in C$, $v_1, v_2, v_3 \in c$ and $\text{lin}(v_1, v_2, v_3)$, then $v_1 = v_2$ or $v_1 = v_3$ or $v_2 = v_3$.
- (v) if two cells with equal sets of vertices are identical.

(2.3) Let $\mathcal{C} = \{(V, \text{lin}), E, C\}$ be a complex. If $\langle v_1, v_2 \rangle \in E$, then v_1 is called the beginning and v_2 is the end of the edge $\langle v_1, v_2 \rangle$. Both v_1, v_2 are called vertices of the edge.

If $c \in C$; $v_1, v_2 \in c$ and v_1 is a predecessor of v_2 with respect to the cyclic order in c , then $\langle v_1, v_2 \rangle$ is called an edge of the cell c . Any $v \in c$ is called a vertex of c . The set of all vertices of a cell c is denoted by $\text{vert}(c)$, and the set of all edges is denoted by $\text{ed}(c)$.

(2.4) For any set A , let $\mathcal{Z}(A)$ be the free abelian group with A as the set of its free generators.

Let $\mathcal{C} = \{(V, \text{lin}), E, C\}$ be a complex and let $c \in C$. Let B_c be the subgroup of $\mathcal{Z}(E)$ generated by all elements of the following two types:

- (a) $\langle v_1, v_2 \rangle + \langle v_2, v_1 \rangle$, for any edge $\langle v_1, v_2 \rangle \in E$,
- (b) $\langle v_1, v_2 \rangle + \langle v_2, v_3 \rangle + \langle v_3, v_1 \rangle$, for any $v_1, v_2, v_3 \in V$ such that $\{v_1, v_2\}$ is an edge of c , $\text{lin}(v_1, v_2, v_3)$ and $\langle v_1, v_2 \rangle, \langle v_2, v_3 \rangle, \langle v_3, v_1 \rangle \in E$.

Denote $\mathcal{Z}(E)/B_c$ by $\mathcal{Z}_c(E)$. We are going to identify each edge $\langle v_1, v_2 \rangle$ with its image in $\mathcal{Z}_c(E)$. Hence we shall write $\langle v_1, v_2 \rangle = -\langle v_2, v_1 \rangle$.

(2.5) For any $c' \in C$, let $\delta(c')$ be the element of $\mathcal{Z}(E)$ equal to the sum of all edges of c' . If there is no danger of ambiguity the image of $\delta(c')$ in $\mathcal{Z}_c(E)$ under the canonical map $\mathcal{Z}(E) \rightarrow \mathcal{Z}(E)/B_c = \mathcal{Z}_c(E)$, where c is a fixed cell, will also be denoted by $\delta(c')$.

(2.6) DEFINITION. Let \mathcal{C} be a complex and let c be one of its cells. Then a family of cells $\{c_1, \dots, c_n\}$ of cells is called a subdivision of the cell c if $\delta c = \sum_{i=1}^n \delta c_i$ in $\mathcal{Z}_c(E)$.

(2.7) EXAMPLE. Consider \mathcal{R}^2 with its canonical orientation determined by the choice of the basis (e_1, e_2) , where $e_1 = (1, 0)$, $e_2 = (0, 1)$.

Let \tilde{c} be a compact, two-dimensional convex subset of \mathcal{R}^2 with a finite number of vertices. Denote by $\text{vert}(\tilde{c})$ the set of all vertices of \tilde{c} cyclically ordered by

if v_1, v_2 are adjacent vertices of \tilde{c} , then v_1 is the predecessor of v_2 iff for any $p \in \text{Int}(\tilde{c})$, the basis $(\overrightarrow{v_1 p}, \overrightarrow{v_1 v_2})$ of \mathcal{R}^2 has the same orientation as (e_1, e_2) .

Denote by $\text{ed}(\tilde{c})$ the set of all pairs $\langle v_1, v_2 \rangle$ where $v_1, v_2 \in \text{vert}(\tilde{c})$ and v_1 is the predecessor of v_2 .

Now let \tilde{C} be any family of compact, two-dimensional convex subsets of \mathcal{R}^2 such that each has a finite number of vertices. Then the system $\mathcal{C}(\tilde{C}) = \{(V(\tilde{C}), \text{lin}), E(\tilde{C}), C(\tilde{C})\}$, where

- (a) $V(\tilde{C}) = \bigcup_{\tilde{c} \in \tilde{C}} \text{vert}(\tilde{c})$,
- (b) $\text{lin}(v_1, v_2, v_3)$ iff v_1, v_2, v_3 lie on the same line in \mathcal{R}^2 .
- (c) $E(\tilde{C}) = \{\langle v_1, v_2 \rangle \in V \times V \mid \langle v_1, v_2 \rangle \text{ or } \langle v_2, v_1 \rangle \in \bigcup_{\tilde{c} \in \tilde{C}} \text{ed}(\tilde{c})\}$,
- (d) $C(\tilde{C})$ is the set of all cyclically ordered sets $\text{vert}(\tilde{c})$, where $\tilde{c} \in \tilde{C}$,

is a complex.

(2.7.1) It follows by a Euclidean geometry argument that:

if a family of cells $\{\text{vert}(\tilde{c}_1), \dots, \text{vert}(\tilde{c}_n)\}$ in $\mathcal{C}(\tilde{C})$ is a subdivision of a cell $\text{vert}(\tilde{c}_0)$, then

(1) $\tilde{c}_0 = \bigcup_{i=1}^n \tilde{c}_i$ and $\text{Int}(\tilde{c}_i) \cap \text{Int}(\tilde{c}_j) = \emptyset$ for $i \neq j$.

(2a) if $\langle v_i, v_j \rangle$ is an edge of \tilde{c}_k for some $k \in \{1, \dots, n\}$, which is not contained in any line containing an edge of \tilde{c}_0 , then there exists exactly one cell \tilde{c}_l , $l \in \{1, \dots, n\}$ and $l \neq k$ such that $\langle v_i, v_j \rangle$ is an edge of \tilde{c}_l ,

(2b) if $\langle v_i, v_j \rangle$ is an edge of \tilde{c}_0 and $\{\langle v_{p_i}, v_{q_i} \rangle \mid 1 \leq i \leq m\}$ is the set of all edges of cells $\tilde{c}_1, \dots, \tilde{c}_n$ that are contained in the line passing through v_i and v_j , then there exists a permutation σ of $\{1, \dots, m\}$ such that

- (i) $v_{p_{\sigma(1)}} = v_i$, $v_{q_{\sigma(m)}} = v_j$,
- (ii) $v_{q_{\sigma(i)}} = v_{p_{\sigma(i+1)}}$ for $i = 1, \dots, m-1$.

(2.8) Let X be a compact Kähler manifold with a meromorphic action of $T = \mathbb{C}^* \times \mathbb{C}^*$ (§0.2). Let $\mathcal{C}(X) = \{(V(X), \text{lin}), E(X), C(X)\}$ be a system defined in the following way:

(a) $V(X) = \{F_1, \dots, F_r\}$, where F_1, F_2, \dots, F_r are all connected components of X^T ,

(b) $E(X) \subset V(X) \times V(X)$ and $\langle F_i, F_j \rangle \in E(X)$ iff there exists a one-dimensional orbit Tx , where $x \in X$, such that $\overline{Tx} \cap F_i \neq \emptyset$, $\overline{Tx} \cap F_j \neq \emptyset$, and $\overline{Tx} \cap F_i \cap F_j = \emptyset$,

(c) lin is a three-argument relation in $V(X)$ defined as follows: $\text{lin}(F_i, F_j, F_k)$ iff there exists a one-parameter subgroup $\alpha \in \chi^*(T)$, $\alpha \neq 0$, and a connected component F of X^α such that $F_i \cup F_j \cup F_k \subset F$,

(d) $C(X)$ is the family of all subsets of $V(X)$ which are of the form $\{F_i; \overline{Tx} \cap F_i \neq \emptyset\}$ for all two-dimensional orbits Tx in X , with the cyclic order defined in the following way:

if $F_k, F_l \in \{F_i | \overline{Tx} \cap F_i \neq \emptyset\}$, $F_k \neq F_l$, then F_k is the predecessor of F_l iff there exist one-parameter subgroups $\alpha, \beta \in \chi^*(T)$ such that:

(i) the source of the action of C^* on \overline{Tx} induced by α contains $(F_k \cap \overline{Tx}) \cup (F_l \cap \overline{Tx})$,

(ii) there exists $y \in \overline{Tx}$ such that $\lim_{t \rightarrow 0} \beta(t)y \in F_k$, $\lim_{t \rightarrow \infty} \beta(t)y \in F_l$,

(iii) (α, β) is a basis of $\chi_{\mathcal{A}}^*(T)$ oriented consistently with the canonical basis (e_1, e_2) .

In order to see that the notion of the predecessor defined in (d) determines a cyclic order in $\{F_i | F_i \cap \overline{Tx} \neq \emptyset\}$ consider a moment function $\not\prec: X \rightarrow \chi_{\mathcal{A}}(T)$. The restriction of the moment function to \overline{Tx} has the following properties (see (0.5.1), (0.5.2) and (0.5.3)):

(1) $\not\prec(\overline{Tx}) = \text{conv}(\not\prec(\{F_i | F_i \cap \overline{Tx} \neq \emptyset\}))$ and $\not\prec(\{F_i | F_i \cap \overline{Tx} \neq \emptyset\})$ coincides with the set of vertices of $\not\prec(\overline{Tx})$,

(2) if $F_k, F_l \in \{F_i | F_i \cap \overline{Tx} \neq \emptyset\}$, $F_k \neq F_l$, and there exists a nontrivial $\gamma \in \chi^*(T)$ such that $F_k \cup F_l$ is contained in a connected component F of X^γ , then $\gamma^* \circ \not\prec(F_k) = \gamma^* \circ \not\prec(F_l)$ and $\not\prec(F \cap \overline{Tx}) = \text{conv}(\not\prec(F_k), \not\prec(F_l))$ is an edge of $\not\prec(\overline{Tx})$,

(3) for any $\gamma \in \chi^*(T)$ and any $y \in \overline{Tx} - (\overline{Tx})^T$, if $\gamma \neq 0$, then

$$\gamma^* \circ \not\prec\left(\lim_{t \rightarrow 0} \gamma(t)y\right) < \gamma^* \circ \not\prec\left(\lim_{t \rightarrow \infty} \gamma(t)y\right).$$

Thus conditions (i) and (ii) are equivalent to:

$$(i') \alpha^* \circ \not\prec(F_k) = \alpha^* \circ \not\prec(F_l) = \min \alpha^* \circ \not\prec(\not\prec(\overline{Tx})),$$

$$(ii') \beta^* \circ \not\prec(F_k) < \beta^* \circ \not\prec(F_l).$$

Moreover, condition (iii) is equivalent to:

(iii') the basis of $\chi_{\mathcal{A}}(T)$ dual to the basis (α, β) of $\chi_{\mathcal{A}}^*$ is oriented consistently with the canonical basis of $\chi_{\mathcal{A}}(T)$. Now take any point $p \in \text{Int } \not\prec(\overline{Tx})$ and consider

$$\alpha' = \overrightarrow{\not\prec(F_k)p}, \quad \beta' = \overrightarrow{\not\prec(F_k)\not\prec(F_l)}.$$

Then

$$\alpha^*(\alpha') = \alpha^*(p) - \alpha^*(\not\prec(F_k)) > 0,$$

$$\alpha^*(\beta') = \alpha^*(\not\prec(F_l)) - \alpha^*(\not\prec(F_k)) = 0,$$

$$\beta^*(\beta') = \beta^*(\not\prec(F_l)) - \beta^*(\not\prec(F_k)) > 0.$$

Thus (α', β') is a base of $\chi_{\mathcal{A}}(T)$ oriented consistently with the basis dual to (α, β) and therefore we have proved that

(*) F_k is a predecessor of F_l in the sense of (2.8)(d) iff $\not\prec(F_k)$ and $\not\prec(F_l)$ belong to the same edge of $\not\prec(\overline{Tx})$ and, for any $p \in \text{Int}(\not\prec(\overline{Tx}))$, the base $\overrightarrow{\not\prec(F_k)p, \not\prec(F_k)\not\prec(F_l)}$ of $\chi_{\mathcal{A}}(T)$ is oriented consistently with the canonical base of $\chi_{\mathcal{A}}(T)$.

Therefore the relation of the predecessor defined as in (d) defines a cyclic order (compare with (2.7)).

(2.9) THEOREM. *The system $\mathcal{C}(X)$ defined in (2.8) is a complex.*

PROOF. We are going to check that conditions (i)–(v) of (2.2) are satisfied. Let us fix a moment function $\notag: X \rightarrow \chi_{\mathcal{A}}(T)$.

(i) Let Tx be a one-dimensional orbit in X and $F_i \cap \overline{Tx} \neq \emptyset$, $F_j \cap \overline{Tx} \neq \emptyset$ and $F_i \cap F_j \cap \overline{Tx} = \emptyset$. Let β be any one-parameter subgroups not contained in the isotropy group of x . Then either $\lim_{t \rightarrow 0} \beta(t)x \in F_i$ and $\lim_{t \rightarrow \infty} \beta(t)x \in F_j$ or $\lim_{t \rightarrow 0} \beta(t)x \in F_j$ and $\lim_{t \rightarrow \infty} \beta(t)x \in F_i$. In both cases (by (0.5.2)) $\beta^* \circ \notag(F_i) \neq \beta^* \circ \notag(F_j)$. Thus $F_i \neq F_j$.

(ii) follows immediately from the definition of $E(X)$.

(iii) Let $c = \{F_i | F_i \cap \overline{Tx} \neq \emptyset\}$, where $x \in X$, be a cell and assume that $F_k \in c$ is the predecessor of $F_l \in c$. Then $F_k \neq F_l$ and there exists a one-parameter subgroup $\alpha \in \chi^*(T)$, $\alpha \neq 0$, such that the source X_1 of the induced by α action of \mathbf{C}^* on \overline{Tx} contains $(F_k \cap \overline{Tx}) \cup (F_l \cap \overline{Tx})$. Since the source is connected (see [2, (A.1)]), it has to be of dimension one. Since the source is irreducible, T -invariant and closed, it has to be of the form \overline{Ty} for some $y \in X_1$. Then the edges determined by \overline{Ty} (see (c) in (2.8)) are $\langle F_k, F_l \rangle$ and $\langle F_l, F_k \rangle$.

(iv) If $F_j, F_k, F_l \in c = \{F_i | F_i \cap \overline{Tx} \neq \emptyset\}$ for some $x \in X$ with $\dim Tx = 2$, and $\text{lin}(F_j, F_k, F_l)$, then $F_j \cup F_k \cup F_l$ is contained in a connected component of X^α for some $\alpha \in \chi^*(T)$, $\alpha \neq 0$. Thus $F_k = F_j$ or $F_l = F_k$ or $F_j = F_l$ by (0.5.3).

(v) Assume that $\{F_i | F_i \cap \overline{Tx}_1 \neq \emptyset\} = \{F_i | F_i \cap \overline{Tx}_2 \neq \emptyset\}$ for $x_1, x_2 \in X$ with $\dim Tx_1 = 2 = \dim Tx_2$. Then

$$\begin{aligned} \notag(\overline{Tx}_1) &= \text{conv}(\notag(\{F_i | F_i \cap \overline{Tx}_1 \neq \emptyset\})) \\ &= \text{conv}(\notag(\{F_i | F_i \cap \overline{Tx}_2 \neq \emptyset\})) = \notag(\overline{Tx}_2) \end{aligned}$$

and it follows from (*) of (2.8) that the cyclic orders on both cells, $\{F_i | F_i \cap \overline{Tx}_1 \neq \emptyset\}$, $\{F_i | F_i \cap \overline{Tx}_2 \neq \emptyset\}$, are identical. Hence the cells are equal. \square

(2.9.1) Let c be a cell in $\mathcal{C}(X)$. Let $g: X \rightarrow Y = X/K$ be the quotient map. Any orbit Ay , where $y \in Y$ and $\{F_i | F_i \cap \overline{Ay} \neq \emptyset\} = \text{vert}(c)$, is called a *realization* of c .

Notice that for any $x \in g^{-1}(y)$, where y is as above.

$$\{F_i | F_i \cap \overline{Ay} \neq \emptyset\} = \{F_i | F_i \cap \overline{Tx} \neq \emptyset\}.$$

If $\{c_1, \dots, c_n\}$ is a subdivision of c , then we say that $\bigcup_i Ay_i \subset Y$ is a realization of the subdivision if Ay_i is a realization of c_i for every $i = 1, \dots, n$.

(2.9.2) Let c be a cell in $\mathcal{C}(X)$, let Ay be a realization of c and let $x \in g^{-1}(y)$. Let

$$\begin{array}{ccc} Z_x & \xrightarrow{\phi_x} & X \\ f_x \downarrow & & \\ Q_x & & \end{array}$$

be as in (0.2.2) and let $q \in Q_x$.

Suppose that $\{Tz_1, \dots, Tz_n\}$ is the set of all two-dimensional orbits in $Z_x(q)$ and denote by c_i the cell in $\mathcal{C}(X)$ defined by

$$c_i = \{F_i | F_i \cap T\phi_x(z_i) \neq \emptyset\}.$$

Then $\{c_i, \dots, c_n\}$ is a subdivision of c and $\bigcup \text{Ag}(\phi_x(z_i))$ is a realization of the subdivision.

The proof of this follows from (0.5.1).

(2.10) LEMMA. *Let $\langle F_i, F_j \rangle$ be an edge in $\mathcal{C}(X)$ and let $\alpha, \beta \in \chi^*(T)$, $\alpha \neq 0$, $\beta \neq 0$. Suppose that $F_i \cup F_j \subset F_\alpha \cap F_\beta$, where F_α, F_β are connected components of X^α, X^β , respectively. Then α, β are linearly dependent. $X^\alpha = X^\beta$ and $F_\alpha = F_\beta$.*

PROOF. It is enough to show that α, β are linearly dependent. Let $\mathcal{J}: X \rightarrow \chi_{\mathcal{A}}(T)$ be a moment function. It follows from our assumptions that $\alpha^* \circ \mathcal{J}(F_i) = \alpha^* \circ \mathcal{J}(F_j)$ (since $\alpha^* \circ \mathcal{J}$ is constant on F_α) and similarly $\beta^* \circ \mathcal{J}(F_i) = \beta^* \circ \mathcal{J}(F_j)$. If α, β were linearly independent, then $\alpha_1^* \circ \mathcal{J}(F_i) = \alpha_1^* \circ \mathcal{J}(F_j)$ for any $\alpha_1 \in \chi^*(T)$ and hence $\mathcal{J}(F_i) = \mathcal{J}(F_j)$. But the last equality contradicts our assumption that F_i, F_j are different vertices of the same edge. Thus α, β are linearly dependent. \square

(2.11) It follows from (2.10) that the set of all edges can be divided into (a finite number of) disjoint subsets $E_F(X)$, where F runs over the set of all connected components of subspaces X^α for $\alpha \in \chi^*(T)$, $\alpha \neq 0$, and $\{F_i, F_j\} \in E_F(X)$ iff $F_i \cup F_j \subset F$.

(2.12) Let c_0 be a cell of $\mathcal{C}(X)$ and let $\{c_1, \dots, c_n\}$ be a subdivision of c_0 .

(2.12a) Let $\langle F_i, F_j \rangle$ be an edge of c_k for some $k \in \{1, \dots, n\}$, such that $\langle F_i, F_j \rangle \in E_F(X)$ but no edge of c_0 belongs to $E_F(X)$. Then there exists exactly one cell c_l , $l = \{1, \dots, n\}$ and $l \neq k$, such that $\langle F_j, F_i \rangle$ is an edge of c_l .

(2.12b) Let $\langle F_i, F_j \rangle$ be an edge of c_0 and let $\langle F_i, F_j \rangle \in E_F(X)$. Let $\{\langle F_{p_i}, F_{q_i} \rangle | i = 1, \dots, m\}$ be the set of all edges of cells $\{c_1, \dots, c_n\}$ belonging to $E_F(X)$. Then there exists a permutation σ of $\{1, \dots, m\}$ such that

- (i) $F_{p_{\sigma(1)}} = F_i, F_{q_{\sigma(m)}} = F_j$, and
- (ii) $F_{q_{\sigma(i)}} = F_{p_{\sigma(i+1)}}$ for $i = 1, \dots, m-1$.

PROOF. Let $\mathcal{J}: X \rightarrow \chi_{\mathcal{A}}(T)$ be a moment function and let c be any cell in $\mathcal{C}(X)$. Then \mathcal{J} maps $\text{vert}(c)$ onto $\text{vert}(\text{conv}(\mathcal{J}(\text{vert}(c))))$ (see (0.5)) and \mathcal{J} preserves cyclic orders given on both sets (2.8) and (2.7).

Moreover if $\text{lin}(F_i, F_j, F_k)$ and $\langle F_i, F_j \rangle$ is an edge of c , then $\mathcal{J}(F_i), \mathcal{J}(F_j), \mathcal{J}(F_k)$ lie on the same line containing the edge $\langle \mathcal{J}(F_i), \mathcal{J}(F_j) \rangle$ of $\text{conv}(\mathcal{J}(\text{vert}(c)))$. Therefore the equality $\delta(c_0) = \sum_{i=1}^n \delta(c_i)$ in $\mathcal{Z}_{c_0}(E(X))$ implies that

$$\delta(\mathcal{J}(c_0)) = \sum_{i=1}^n \delta(\mathcal{J}(c_i)) \quad \text{in } \mathcal{Z}_{\mathcal{J}(c_0)}(E(\tilde{C})),$$

where for any cell c in $\mathcal{C}(X)$, $\mathcal{J}(c) = \mathcal{J}(\text{vert}(c))$ and $\tilde{C} = \{\text{conv} \mathcal{J}(c_0), \text{conv} \mathcal{J}(c_1), \dots, \text{conv} \mathcal{J}(c_n)\}$ (compare (2.7)). Thus $\{\mathcal{J}(c_1), \dots, \mathcal{J}(c_n)\}$ is a subdivision of $\mathcal{J}(c_0)$ in $\mathcal{C}(\tilde{C})$ and (2.12) follows from (2.7.1). \square

(2.13) Let $\mathcal{C} = \{(V, \text{lin}), E, C\}$ be a complex. A map $\mu: C \rightarrow \{0, 1\}$ is said to be a *moment measure* on \mathcal{C} if for any cell $c_0 \in C$ and any subdivision $\{c_1, \dots, c_n\}$ of c_0 .

$$\mu(c_0) = \sum_{i=1}^n \mu(c_i).$$

If μ is a moment measure on \mathcal{C} , then a cell c is called μ -stable if $\mu(c) = 1$.

(2.13.1) Let μ be a moment measure on $\mathcal{C}(X)$. A point $x \in X$ is said to be μ -stable if $\text{Ag}(x)$ is a realization of a μ -stable cell. The set of all μ -stable points is denoted by X_μ^s .

(2.14) A cell $c \in \mathcal{C}(X)$ is called *principal* (or *generic*) if it has a realization Tx , where the orbit Tx is generic in the sense that $\phi_x(Z_x) = X$.

It follows that there exists exactly one principal cell in $\mathcal{C}(X)$.

(2.14.1) LEMMA. *If F_p is a vertex of a principal cell c of $\mathcal{C}(X)$, then there exists a one-parameter subgroup $\beta \in \chi^*(T)$ such that $X^\beta = X^T$ and F_p is the sink of the induced by β action of \mathbf{C}^* on X .*

PROOF. Let $f: X \rightarrow \chi_{\mathcal{C}}(T)$ be a moment function. It follows from (0.5) that for any generic orbit Tx , $f(Tx) = f(X)$. Hence (again by (0.5)) for any vertex F_p of the principal cell c , $f(F_p)$ is a vertex of $f(X)$. But for any vertex F_p of $f(X)$ there exists a one-parameter subgroup β_p such that $X^{\beta_p} = X^T$ and $\beta_p^* \circ f(F_p)$ is the maximum $\beta_p \circ f(X)$. Then it follows from (0.5.2) that F_p is the sink of the induced by β_p action of \mathbf{C}^* on X . \square

3. Main Theorem.

(3.1) MAIN THEOREM. *Let $T \times X \rightarrow X$ be a meromorphic action of $T \approx \mathbf{C}^* \times \mathbf{C}^*$ on a compact connected Kähler manifold X . Then there is a one-to-one correspondence between moment measures μ and T -invariant open sets $U \subseteq X$ such that the geometric quotient U/T exists and is compact. Such U are automatically Zariski open. The correspondence is given by sending μ to the μ stable points X_μ^s .*

PROOF. Assume we have a moment measure μ . Then $X - U$ is a closed analytic set and U is Zariski open in X . To see this, note that given x with $\dim Tx = 2$, then the family of cells associated to any $\phi_x(Z_x(q))$ with $q \in Q_x$ is a subdivision of the cell associated to \overline{Tx} . Thus if $\mu(\overline{Tx}) = 0$ it follows that $\phi_x(Z_x) \subseteq X - U$. Thus $X - U$ is the union of the $\phi_x(Z_x)$ with $\mu(\overline{Tx}) = 0$ and all $x \in X$ such that $\dim Tx \leq 1$. Thus by (0.2.1) and (0.2.4) it follows that $X - U$ is the union of finitely many closed analytic sets. Therefore $X - U$ is a closed analytic set and U is Zariski open in X .

The set U is trivially T -invariant. If $\mu(c) = 1$, then given any subdivision $\{c_1, \dots, c_n\}$ of c we get precisely one c_i with $\mu(c_i) = 1$. Thus it follows from (0.5.1)(b) that for each $q \in Q$, $\phi(Z(q)) \cap U = Tx$ for some x such that $\dim Tx = 2$. By (0.4) U/T exists and is compact.

(3.2) Note that if we show that a T -invariant open U with a compact geometric quotient U/T is the set of μ stable points X_μ^s associated to a moment measure μ , then by the above, U is Zariski open in X .

It remains to show that for any open T -invariant subset $U \subset X$ such that the geometric quotient U/T exists and is compact there exists a moment measure μ such that $U = X_\mu^s$.

First we fix notation and terminology used in this part of the paper:

(a) all chains and cochains considered are assumed to be oriented singular cell chains and cochains with rational coefficients,

(b) since the quotient map $g: X \rightarrow X/K = Y$ restricted to any fixed point component F_i of X^T is a homeomorphism onto its image we are going to identify F_i with $g(F_i)$ and consider F_i as a subspace of Y ,

(c) $X_1(Y_1)$ denotes $\{X \in X | \dim Tx \leq 1\}$ ($\{y \in Y | \dim Ay \leq 1\}$, resp.),

(d) if $y \in Y$ and $\dim Ay = 2$, then we are going to consider \overline{Ay} both as a subspace of Y and as a 2-chain in Y . (This is possible since any moment function $\not\prec: X \rightarrow \chi_{\mathcal{A}}(T)$ determines a homeomorphism of \overline{Ay} onto a compact convex two-dimensional subset of $\chi_{\mathcal{A}}(T)$ (see (0.5)(b)) and the orientation of the subset does not depend on the choice of $\not\prec$ (see (2.8))),

(e) if α is a one-parameter subgroup $\alpha \in \chi^*(T)$, then Y^α denotes $g(\chi^\alpha)$.

Now we prove two lemmas.

(3.3) LEMMA. *Let F_p be any vertex of the principal cell of $\mathcal{C}(X)$. Let $\alpha \in \chi^*(T)$, $\alpha \neq 0$, and let F_α be a connected component of X^α . For any given 1-cycle a in F_α there exists a 2-chain b in X_1 such that $\delta(b) = a + a_1$, where a_1 is a 1-cycle in F_p .*

PROOF. Since F_p is a vertex of the principal cell, there exists a one-parameter subgroup $\beta \in \chi^*(T)$ such that $X^\beta = X^T$ and F_p is the sink of the induced β action of C^* on X (see (2.14.1)). We are going to construct:

(a) a sequence $\{\alpha_0, \alpha_1, \dots, \alpha_m\}$ of one-parameter subgroups of T ,

(b) a sequence $\{W_0, W_1, \dots, W_m\}$ of submanifolds of X satisfying the following properties:

(i) W_i is a connected component of X^{α_i} for $i = 0, \dots, m$,

(ii) $\alpha_0 = \alpha$, $W_0 = F_\alpha$,

(iii) the sink $F_{l(i)}$ of the induced β action of C^* on W_i is contained in W_{i+1} and is different from the sink $F_{l(i+1)}$ of the β action of C^* on W_{i+1} .

(iv) the sink of the induced β action of C^* on W_m is equal to F_p .

Let us put $\alpha_0 = \alpha$, $W_0 = F_\alpha$. Then assume that the sequences $\{\alpha_0, \dots, \alpha_k\}$, $\{W_0, \dots, W_k\}$ satisfying (i)–(iii) have already been constructed for some $k \geq 0$. If the sink of the action of C^* on W_k is F_p , the construction is finished. Suppose that the sink $F_{l(k)}$ of the action on W_k is not F_p . Then it follows from our assumption that $F_{l(k)}^+ \neq F_{l(k)}$, where $F_{l(k)}^+$ is defined by the (induced by β) action of C^* on X (see [4]). It follows from [4] (see also [2] for the algebraic case) that the map $\pi: F_{l(k)}^+ \rightarrow F_{l(k)}$, defined by $\pi(x) = \lim_{t \rightarrow 0} \beta(t)x$ for any $x \in F_{l(k)}^+$, determines a locally trivial space with affine spaces as fibres. Moreover the fibres are T -invariant and the action of T on any fibre is linear. Take any point $y \in F_{l(k)}$. Since $\pi^{-1}(y)$ has the structure of a linear space and the restriction of the action of T to $\pi^{-1}(y)$ is linear and nontrivial, there exists a nontrivial character $\chi \in \chi(T)$ and a point $x \in \pi^{-1}(y) - \{y\}$ such that for any $t \in T$, $T(x) = \chi(t) \cdot x$ (where the multiplication of $\chi(t)$ times x is

taken with respect to the linear space structure of $\pi^{-1}(y)$). Now let α' be any nontrivial one-parameter subgroup of T such that $(\chi, \alpha') = 0$. Then $X^{\alpha'} \supset F_{l(k)} \cup \overline{Tx}$. Since $\overline{Tx} \cap F_{l(k)} = \{y\} \neq \emptyset$, the connected component F' of $X^{\alpha'}$ containing $F_{l(k)}$ contains x and therefore the sink of the induced β action of \mathbf{C}^* on F' is different from $F_{l(k)}$. Let us put $\alpha_{k+1} = \alpha'$, $W_{k+1} = F'$.

Let $\mathcal{J}: X \rightarrow \mathcal{R}^2$ be any moment function. Then it follows from (0.5.1) that if $F_{l(k+1)}$ is the sink of W_{k+1} (with respect to the action of \mathbf{C}^* induced by β), then

$$\beta^* \circ \mathcal{J}(F_{l(k)}) < \beta^* \circ \mathcal{J}(F_{l(k+1)}).$$

Since there exists only a finite number of components F_i , after a finite number of steps our construction of sequences (a), (b) must stop. If it stops after m steps, then (iv) is satisfied and we obtain sequences (a), (b) with properties (i)–(iv).

Now in order to prove the lemma apply (1.8) $(m+1)$ times. First take a 2-chain b_0 in W_0 such that $\delta b_0 = a + a_1$, where a_1 is a 1-cycle in $F_{l(0)}$. Then find a 2-chain b_1 in W_1 such that $\delta(b_1) = a_1 + a_2$, where a_2 is a 1-cycle in $F_{l(1)}$, etc. Finally, take a 2-chain b_m in W_m such that $\delta b_m = a_m + a_{m+1}$, where a_{m+1} is a 1-cycle in $F_{l(m)} = F_p$. Then $b = b_0 - b_1 + \cdots + (-1)^m b_m$ is a 2-chain in Y_1 ,

$$\delta(b) = (a + a_1) - (a_1 + a_2) + \cdots + (-1)^m (a_m + a_{m+1}) = a + (-1)^m a_{m+1},$$

and a_{m+1} is a 1-cycle in F_p . The proof of the lemma is finished. \square

(3.4) COROLLARY. Let F_p be a vertex of the principal cell. Let $\alpha \in \chi^*(T)$, $\alpha \neq 0$, and let F_α be a connected component of Y^α . For any 1-cycle a' in F_α , there exists a 2-chain b' in Y_1 such that $\delta b' = a' + a'_1$, where a'_1 is a 1-cycle in F_p .

PROOF. Since the fibres of the quotient map $g: X \rightarrow X/K = Y$ are pathwise connected, there exists a 1-cycle a in $g^{-1}(F_\alpha)$ such that $g_*(a) = a'$. Since $g^{-1}(F_\alpha)$ is a connected component of X^α we may apply Lemma (3.3) and obtain a 2-chain b in X_1 such that

$$\delta(b) = a + a_1, \quad \text{where } a_1 \text{ is a 1-cycle in } F_p.$$

Then $b' = g_*(b)$ and $a'_1 = g_*(a_1)$ satisfy the conditions of the corollary. \square

(3.5) LEMMA. Let b be a 2-cocycle with compact support in V such that $[b] \in H_c^2(V)$ is different from zero. Let c_0 be a cell which admits a realization Ax_0 such that $x_0 \in V$. Then for any subdivision $\{c_1, \dots, c_n\}$ of c_0 and its realization $\bigcup_{j=1}^n Ax_j$, the Kronecker product $(b, \sum_{j=1}^n \overline{Ax_j})$ is equal to $(b, \overline{Ax_0})$.

PROOF. It suffices to show that $(b, \overline{Ax_0} - \sum_{j=1}^n \overline{Ax_j}) = 0$.

Let F_i be a connected component of $Y^A = X^T$. For any $y \in Y$ such that $\overline{Ay} \cap F_i \neq \emptyset$, the intersection is composed of exactly one point, denote this point by $V_i(y)$. Now, for any x_k, x_l , where $k, l = 0, \dots, n$ such that $\overline{Ax_k} \cap F_i \neq \emptyset$ and $\overline{Ax_l} \cap F_i \neq \emptyset$, choose an arc γ_{kl}^i in F_i with the beginning at $v_i(x_k)$ and the end at $v_i(x_l)$. We may assume that $\gamma_{kl}^i = -\gamma_{lk}^i$.

If $\langle F_i, F_j \rangle$ is an edge of a cell c_l , $l = 0, \dots, n$, then there exists a one-parameter subgroup $\alpha: \mathbf{C}^* \rightarrow T$, $\alpha \neq 0$, and a connected component Z of Y^α such that $F_i \cup F_j \subset Z$. There exists an arc λ'_{ij} in $\overline{Ax_l} \cap Z$ with the beginning at $v_i(x_l)$ and its

end at $v_j(x_l)$. (In fact one may consider λ_{ij}^l as the intersection of the boundary of the 2-chain Ax_l with Z .)

Let I_0 be the family of all two-element sets $\{i, j\}$ such that $\langle F_i, F_j \rangle$ and $\langle F_j, F_i \rangle$ are edges of some cells c_k, c_l , where $k, l = 1, \dots, n$. Then for $\{i, j\} \in I_0$ define

$$e_{ij} = -(\gamma_{lk}^i + \lambda_{ij}^k + \gamma_{kl}^j + \lambda_{ji}^l).$$

Notice that $e_{ij} = -e_{ji}$.

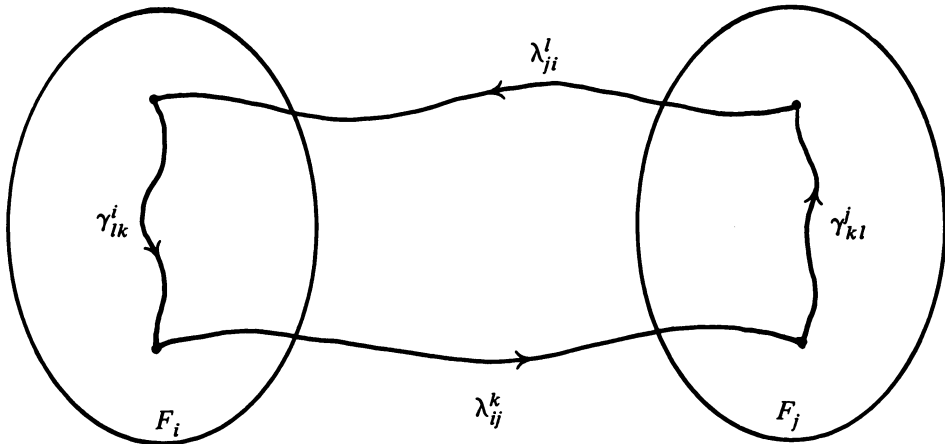


FIGURE 1

Now let I_1 be the set of all pairs $\langle i, j \rangle$ such that $\langle F_i, F_j \rangle$ is an edge of c_0 . Then it follows from (2.12b) that there exists a connected component Z of Y^α , where α is a nontrivial one-parameter subgroup such that $F_i \cup F_j \subset Z$ and there exists a sequence $F_i = F_{i_0}, F_{i_1}, \dots, F_{i_s+1} = F_j$ such that

- (a) $F_{i_j} \subset Z$ for $j = 0, \dots, s$,
- (b) $\langle F_{i_j}, F_{i_{j+1}} \rangle$ is an edge of a cell $c_{k(j)}$, $k(j) \in \{1, \dots, n\}$.

Then for $\langle i, j \rangle \in I_1$ define

$$e_{ij} = \lambda_{ij}^0 + \gamma_{0,k(s)}^j - \lambda_{i_s,i_{s+1}}^{k(s)} + \gamma_{k(s),k(s-1)}^{i_s} + \dots + \gamma_{k(0),0}^{i_1}.$$

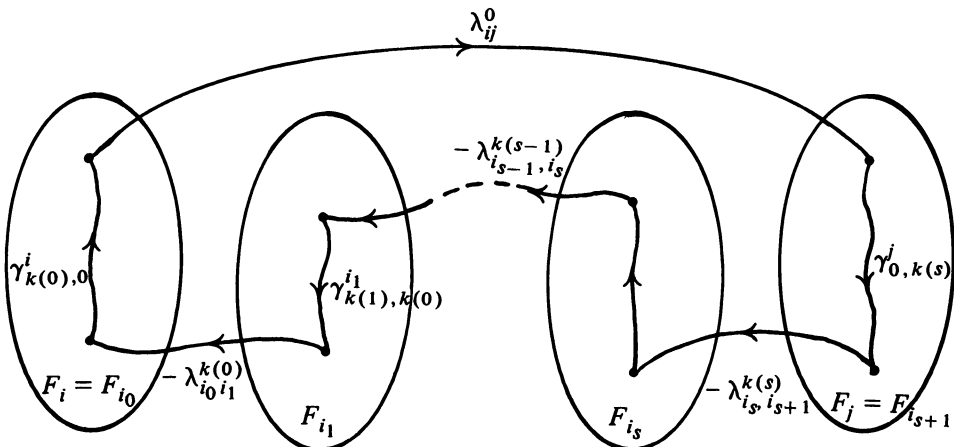


FIGURE 2

Let $I = I_0 \cup I_1$. We have defined above for any $t \in I$ an l -cycle e_t lying in some connected component Z of Y^α for some nontrivial one-parameter subgroup α .

Let F_p be a vertex of a principal cell. It follows from (3.4) that for any $t \in I$ there exists a 2-chain τ_t contained in Y_1 such that $\delta\tau_t = e_t + \mu_t$, where μ_t is an l -cycle contained in F_p .

Since $\delta\overline{Ax_j} = \sum \lambda_{kl}^j$, where the summation runs over all pairs $\langle k, l \rangle$ such that $\langle F_k, F_l \rangle$ is an edge of c_j , the difference $\sum_{t \in I} e_t - (\delta\overline{Ax_0} - \sum_{j=1}^n \delta\overline{Ax_j})$ is a 1-cycle which is a sum of arcs γ_{kl}^i . For fixed $i = 1, \dots, r$ let Σ_i be the sum of those arcs γ_{kl}^i that occur in the difference. Then

(a) Σ_i is a 1-cycle in F_i , $i = 1, \dots, r$,

(b) $\sum_{t \in I} e_t - (\delta\overline{Ax_0} - \sum_{j=1}^n \delta\overline{Ax_j}) = \sum_{i=1}^r \Sigma_i$.

In fact (a) follows from the fact that $\sum_{i=1}^r \Sigma_i$ is a 1-cycle and $F_i \cap F_j = \emptyset$ for $i \neq j$, and (b) is easily checked.

It follows from (3.4) that for F_p chosen as above, and for any Σ_i there exists a 2-chain n_i contained in Y_1 and a 1-cycle m_i contained in F_p such that $\delta(n_i) = \Sigma_i + m_i$.

Since 2-chains τ_t for $t \in I$ and n_i , $i = 1, 2, \dots, r$, are contained in Y_1 and the support of b is in V , we have

$$\left(b, \overline{Ax_0} - \sum_{j=1}^n \overline{Ax_j} \right) = (b, \psi) \quad \text{where } \psi = \overline{Ax_0} - \sum_{j=1}^n \overline{Ax_j} - \sum_{t \in I} \tau_t + \sum_{i=1}^r n_i.$$

On the other hand (by (b))

$$\begin{aligned} \delta\psi &= \delta \left(\overline{Ax_0} - \sum_{j=1}^n \overline{Ax_j} \right) - \sum_{t \in I} (e_t + \mu_t) + \sum_{i=1}^r (\Sigma_i + m_i) \\ &= - \sum_{t \in I} \mu_t + \sum_{i=1}^r m_i. \end{aligned}$$

Thus $\delta\psi$ is a 1-cycle contained in F_p .

Finally notice that since $H_c^2(V) \rightarrow H^2(Y)$ is the zero-homomorphism (see (1.7.1)), $b = \partial b_1$, where b_1 is a 1-cochain in Y and $(b, \psi) = (\partial b_1, \psi) = (b_1, \delta\psi)$.

But $\delta\psi$ is a 1-cycle in F_p that is a boundary of a 2-chain ψ in Y . Thus $\delta\psi$ is also a boundary of a 2-chain ψ' in F_p (see (1.8)). Thus

$$(b_1, \delta\psi) = (b_1, \delta\psi') = (\partial b_1, \psi') = (b, \psi') = 0$$

since the support of b is in V and ψ' is a 2-cycle in F_p .

Therefore $(b, \overline{Ax_0} - \sum_{j=1}^n \overline{Ax_j}) = 0$ and the proof is complete. \square

(3.6) *Proof of the second half of the Main Theorem.* It follows from (1.3) and (1.4) that $H_c^2(U) \approx H_c^2(V) \approx Q$. Choose a 2-cocycle b with compact support in V such that $[b] = 1 \in Q = H_c^2(V)$.

Let c_0 be any cell which admits a realization Ax_0 , where $x_0 \in V$. Let $\{c_1, \dots, c_n\}$ be any subdivision of c_0 and let $\cup Ax_i$ be a realization of the subdivision. Then $\sum Ax_i$ is a 2-chain in Y and it follows from (3.5) that the Kronecker products $(b, \overline{Ax_0})$ are $(b, \sum_{i=1}^n \overline{Ax_i})$ are equal.

Since $(b, \overline{Ax_i}) = 0$ whenever Ax_i is not contained in V ,

$$\left(b, \sum_{j=1}^n \overline{Ax_j}\right) = \sum_{j=1}^n (b, \overline{Ax_j}) = \sum_{Ax_j \in V} (b, \overline{Ax_j}).$$

On the other hand for any $x \in V$ the value of the map

$$H_2(\overline{Ax}, Ax) \xrightarrow{i^*} H_2(\overline{V}, V),$$

where $i: (\overline{Ax}, Ax) \hookrightarrow (\overline{V}, V)$ at $[\overline{Ax}] \in H_2(\overline{Ax}, Ax)$, does not depend on the choice of x . Therefore $(b, \overline{Ax_0}) = (b, \overline{Ax_i})$ for any $x_i \in V$.

Thus there exists exactly one $i \in \{1, \dots, n\}$ such that $x_i \in V$, and the index i does not depend on the choice of the realization.

In particular if Ax'_0 is any realization of c_0 , then $Ax'_0 \subset V$.

Let $\mu: C(X) \rightarrow \{0, 1\}$ be defined as follows:

$$\mu(c) = \begin{cases} 1 & \text{iff for some (any) realization } Ax \text{ of } c, x \in V, \\ 0 & \text{otherwise.} \end{cases}$$

It follows from the above that μ is a moment measure and U is exactly the set of all μ -stable points of X . \square

(3.7) We would like to give a very simple example in order to give the reader a feeling for what the Main Theorem is saying. We intend to study nontrivial examples in another paper—such examples quickly reduce to combinatorial considerations very different from his paper.

Let C^* act on P^2 by

$$(*) \quad (t, [z_0, z_1, z_2]) \rightarrow [z_0, t \cdot z_1, t^2 \cdot z_2],$$

where $t \in C^*$ and $[z_0, z_1, z_2]$ are homogeneous coordinates on P^2 . Let $X = P^2 \times P^2$. Let $T \approx C^* \times C^*$ act on X by sending

$$(t_1, t_2, x_1, x_2) \rightarrow (t_1 \cdot x_1, t_2 \cdot x_2),$$

where $(t_1, t_2) \in C^* \times C^*$, $(x_1, x_2) \in X$, and $t_i \cdot x_i$ for $i = 1, 2$ is given by $(*)$ above.

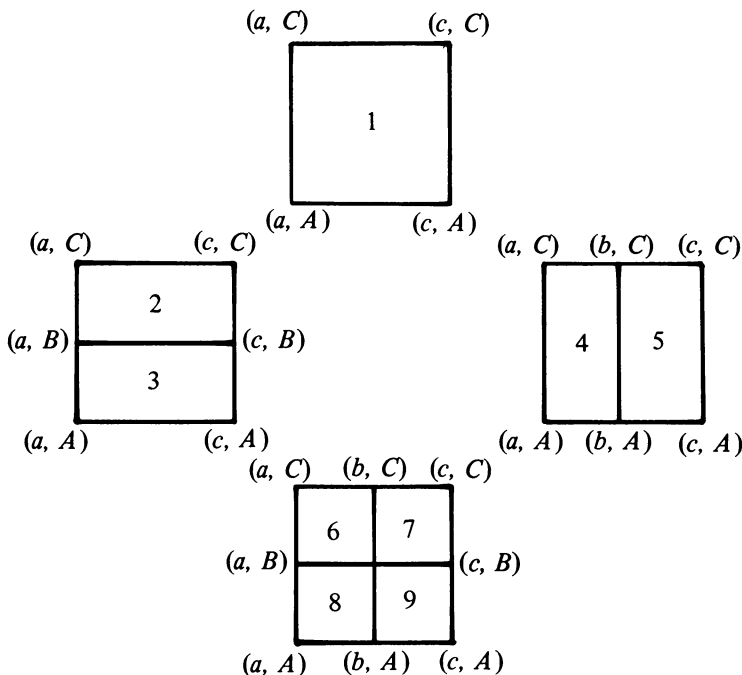
It is easy to check that the above action has nine fixed points

$$\begin{array}{ccc} (a, A) & (a, B) & (a, C) \\ (b, A) & (b, B) & (b, C) \\ (c, A) & (c, B) & (c, C) \end{array}$$

where

$$a = [1, 0, 0] = A, \quad b = [0, 1, 0] = B, \quad c = [0, 0, 1] = C.$$

The nine fixed points above are the vertices of $C(X)$. There are nine 2-cells:



In the above picture the three vertices are on a common line precisely if they satisfy the relation lin of (2.8). Also the edges of the above cells correspond to the edges of (2.8). By the Main Theorem there are four Zariski open T -invariant sets $U \subseteq X$ with a compact geometric quotient under T . To give such a U it suffices to give the cells on which the associated moment measure is 1. The four sets of such 2-cells are

- all the cells containing cell 6, i.e. $\{1, 2, 4, 6\}$,
- all the cells containing cell 7, i.e. $\{1, 2, 5, 7\}$,
- all the cells containing cell 8, i.e. $\{1, 3, 4, 8\}$,
- all the cells containing cell 9, i.e. $\{1, 3, 5, 9\}$.

In the above example the image of any moment map is a rectangle with the nine vertices arranged as in the bottom cell of the above picture.

Though the U 's with compact quotient in the above product example can be computed by the results of [2], the flavour of the above argument is typical for applications of the results of this paper.

(3.8) There is one large class of examples worth mentioning. Let $T \approx \mathbb{C}^{*2}$ act on X . Let $\notag: X \rightarrow \chi_{\notag}(T)$ be a moment map. Let $x \in \chi_{\notag}(T)$ be any point not in the image of any orbit of dimension lower than 2. Let U be the set of all points y in X such that

$$(*) \quad \dim Ty = 2 \quad \text{and} \quad x \in \notag(T \cdot y).$$

Then U has a compact geometric quotient; the 2-cells where the moment measure corresponding to U is equal to 1 are precisely the 2-cells corresponding to the orbits in $(*)$ above.

This class of examples works for C^{**k} if we replace the 2's above by k . We can show that they do not give all examples, even for products of Grassmannians. Though the exact role of these examples is not completely worked out yet, all indications are that for projective varieties they correspond to the examples that Mumford's theory gives. This is currently being actively researched.

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